# ME 8043114 Automatic Control

**Lecture 2: Solution to ODEs via Laplace Transform**
**Modeling Electrical and Mechanical Systems**

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# Lecture Outline

- ODE Solution using Laplace transform
- Modeling electrical and Mechanical Systems

# Course roadmap

### *Modeling*

- Laplace transform
- Transfer function
- Models for systems
  - electrical
  - mechanical
  - electromechanical
- Linearization

### *Analysis*

- Time response
  - Transient
  - Steady state
- Frequency response
  - Bode plot
- Stability
  - Routh-Hurwitz
  - Nyquist

### *Design*

- Design specs
- Root locus
- Frequency domain
- PID & Lead-lag
- Design examples

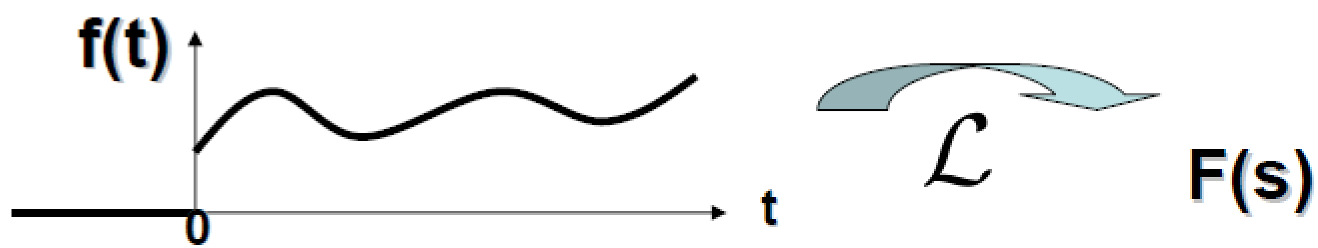
*Matlab simulations & laboratories*

# Laplace transform (review)

- One of most important math tools in the course!
- Definition: For a function f(t) (f(t)=0 for t<0),

$$\boxed{F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt}$$

(s: complex variable)




- We denote Laplace transform of f(t) by F(s).

# Laplace transform table

| $f(t)$ | $F(s)$ |
|---|---|
| $\delta(t)$ | $1$ |
| $u(t)$ | $\frac{1}{s}$ |
| $tu(t)$ | $\frac{1}{s^2}$ |
| $t^n u(t)$ | $\frac{n!}{s^{n+1}}$ |
| $e^{-at}u(t)$ | $\frac{1}{s+a}$ |
| $\sin \omega t \cdot u(t)$ | $\frac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t \cdot u(t)$ | $\frac{s}{s^2+\omega^2}$ |
| $te^{-at}u(t)$ | $\frac{1}{(s+a)^2}$ |

$\mathcal{L}$ $\longrightarrow$

$\mathcal{L}^{-1}$ $\longleftarrow$

Inverse Laplace Transform

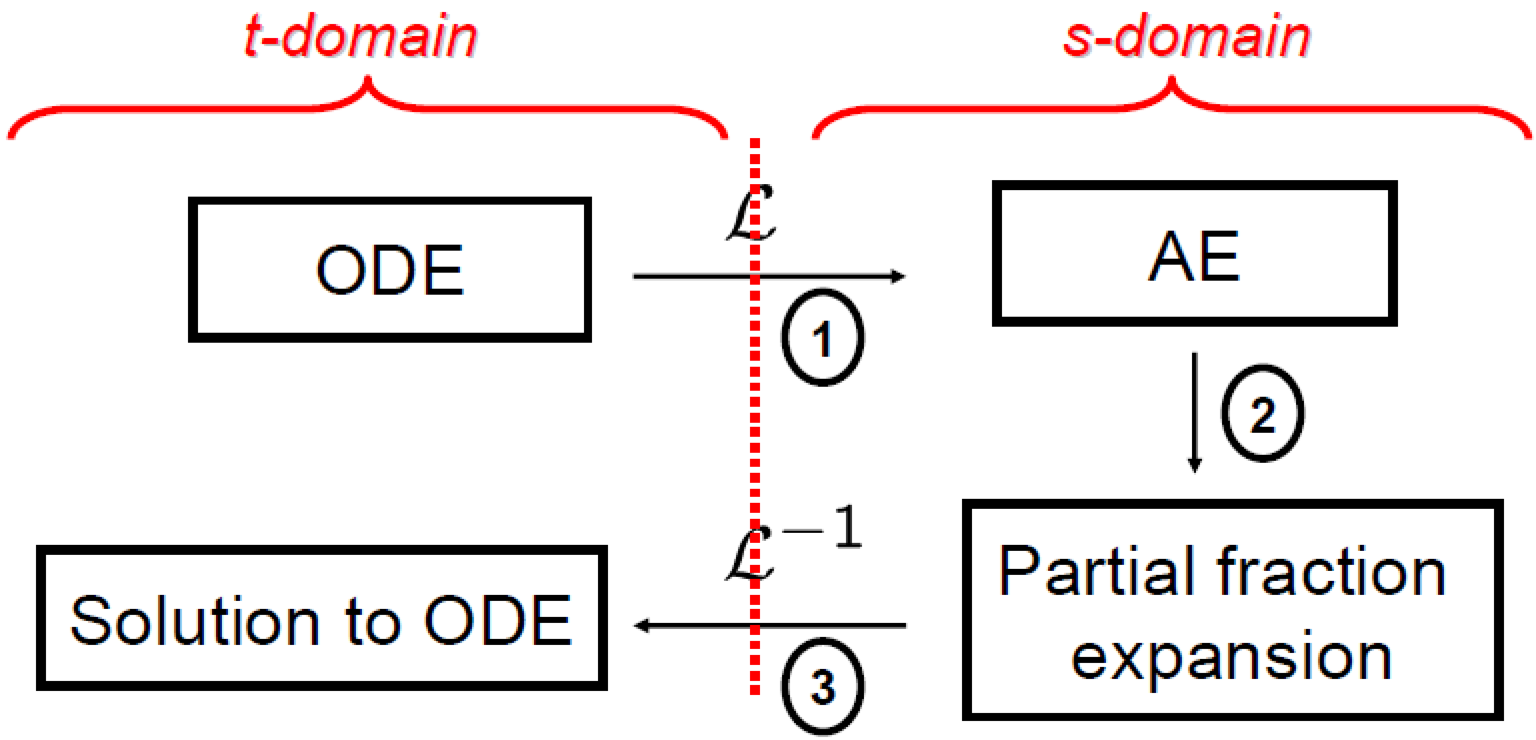
(u(t) is often omitted.)

# Advantages of s-domain (review)

- We can transform an ordinary differential equation into an algebraic equation which is easy to solve. (This lecture)
- It is easy to analyze and design interconnected (series, feedback etc.) systems. (Throughout the course)
- Frequency domain information of signals can be dealt with. (Lectures for frequency responses: after midterm)

# An advantage of Laplace transform

- We can transform an ordinary differential equation (ODE) into an algebraic equation (AE).



# Example 1 (distinct roots)

ODE with initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=5u(t),\ y(0)=-1,\ y'(0)=2$$

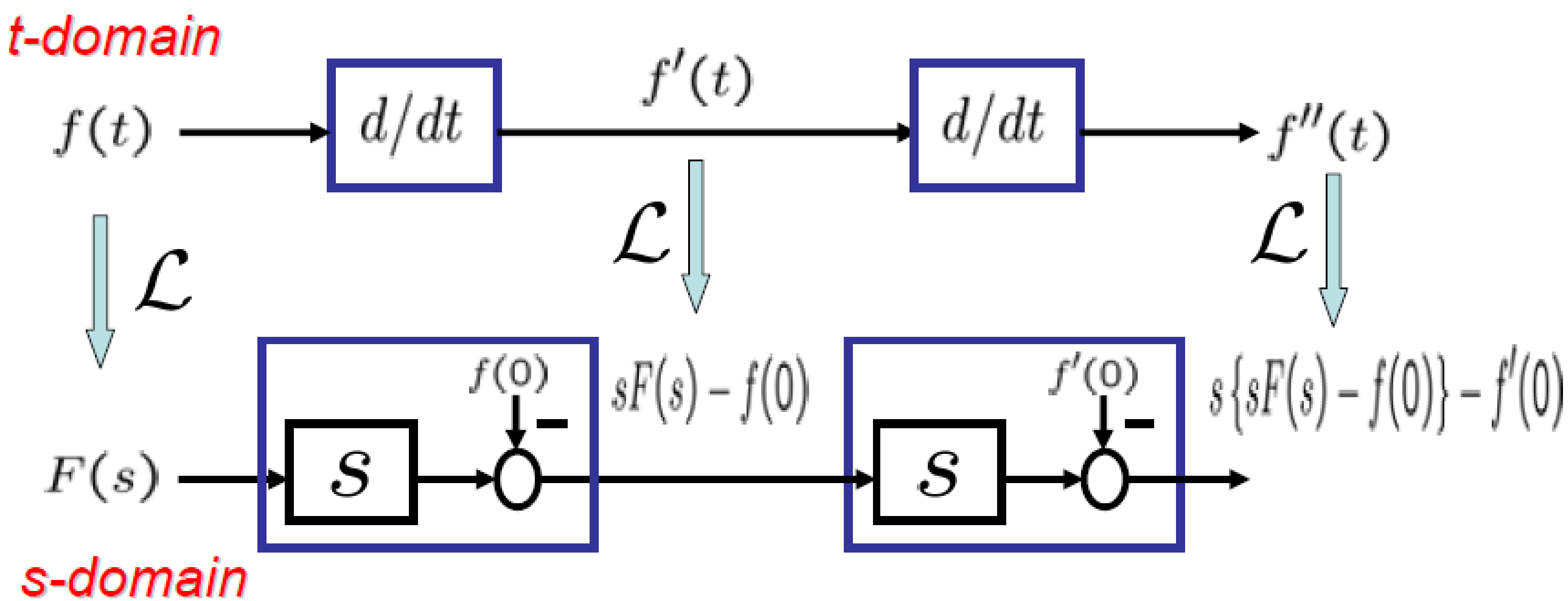
1. Laplace transform

$$\underbrace{s^2Y(s)-sy(0)-y'(0)}_{\mathcal{L}\{y''(t)\}}+3\underbrace{\{sY(s)-y(0)\}}_{\mathcal{L}\{y'(t)\}}+2Y(s)=\frac{5}{s}$$

$$\Rightarrow\quad Y(s)=\frac{-s^2-s+5}{s(s+1)(s+2)}$$ ← *distinct roots*

# Properties of Laplace transform
## *Differentiation (review)*

$$\boxed{\mathcal{L}\left\{f'(t)\right\} = sF(s) - f(0)}$$



# Example 1 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{-s^2 - s + 5}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

Multiply both sides by s & let s go to zero:

$$sY(s)|_{s\to 0} = A + s\frac{B}{s+1}\Big|_{s\to 0} + s\frac{C}{s+2}\Big|_{s\to 0} \Rightarrow A = sY(s)|_{s\to 0} = \frac{5}{2}$$

Similarly,

$$B = (s+1)Y(s)|_{s\to -1} = \cdots = -5$$

$$C = (s+2)Y(s)|_{s\to -2} = \cdots = \frac{3}{2}$$

# Example 1 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$$

$$\Rightarrow \quad y(t) = \left( \underbrace{\frac{5}{2}}_{A} + \underbrace{(-5)}_{B} e^{-t} + \underbrace{\frac{3}{2}}_{C} e^{-2t} \right) u(t)$$

*(You may omit u(t).)*

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{-s^2 - s + 5}{(s+1)(s+2)} = \frac{5}{2}$$

# Example 2 (repeated roots)

ODE with initial conditions (ICs)

$$\frac{d^3y(t)}{dt^3}+5\frac{d^2y(t)}{dt^2}+8\frac{dy(t)}{dt}+4y(t)=2\delta(t),\ y(0)=y'(0)=y''(0)=0$$

1. Laplace transform

$$\begin{aligned}
&s^3Y(s)-s^2y(0)-sy'(0)-y''(0) \longleftarrow \mathcal{L}\left\{y'''(t)\right\}\\
&+5\left\{s^2Y(s)-sy(0)-y'(0)\right\} \longleftarrow 5\mathcal{L}\left\{y''(t)\right\}\\
&+8\left\{sY(s)-y(0)\right\}+4Y(s)\\
&=2
\end{aligned}$$

$$\Rightarrow \quad Y(s)=\frac{2}{(s+1)(s+2)^2}$$

*Repeated roots*

# Example 2 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

To obtain A:

$$(s+1)Y(s)\big|_{s\to -1} = A + (s+1)\frac{B}{(s+2)^2}\bigg|_{s\to -1} + (s+1)\frac{C}{s+2}\bigg|_{s\to -1} \Rightarrow A = 2$$

To obtain B:

$$(s+2)^2 Y(s)\big|_{s\to -2} = (s+2)^2 \frac{A}{s+1}\bigg|_{s\to -2} + B + (s+2)C\big|_{s\to -2} \Rightarrow B = -2$$

# Example 2 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{2}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

To obtain C:

$$\frac{2}{s+1} = (s+2)^2 \frac{A}{s+1} + B + (s+2)C$$

*Take derivative*

$$\frac{-2}{(s+1)^2} = \frac{As(s+2)}{(s+1)^2} + C$$

$$C = -2$$

*Let s go to -2.*

# Example 2 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s+1} + \frac{B}{(s+2)^2} + \frac{C}{s+2}$$

$$\Rightarrow \quad y(t) = \left( \underbrace{2}_{A} e^{-t} + \underbrace{(-2)}_{B} te^{-2t} + \underbrace{-2}_{C} e^{-2t} \right) u(t)$$

If we are interested in only the final value of y(t), apply Final Value Theorem:

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s) = \lim_{s \to 0} \frac{2s}{(s+1)(s+2)^2} = 0$$

# Properties of Laplace transform
## *8. Frequency shift theorem (review)*

$$\boxed{\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)}$$

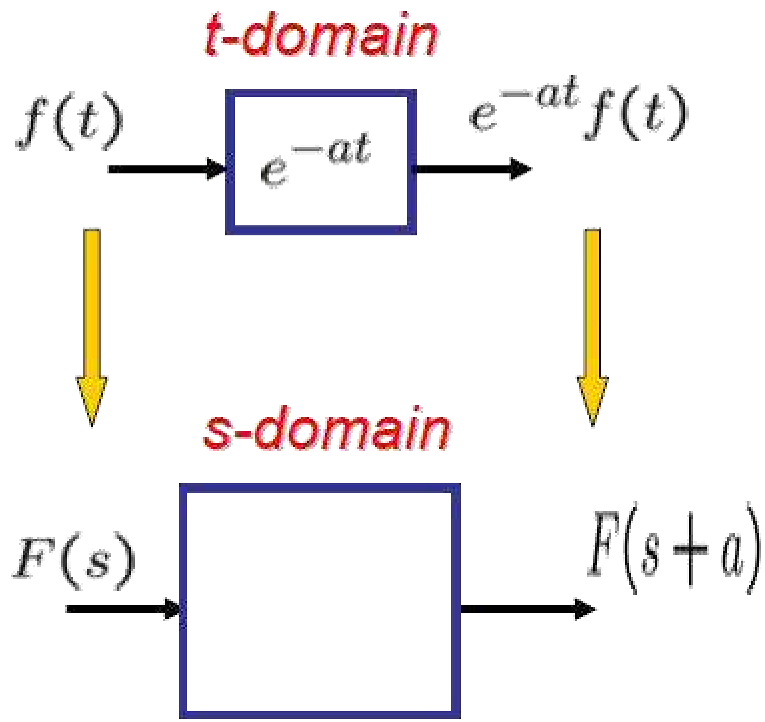
Proof.

$$\mathcal{L}\left\{e^{-at}f(t)\right\} = \int_0^{\infty} e^{-at}f(t)e^{-st}dt$$

$$= \int_0^{\infty} f(t)e^{-(s+a)t}dt = F(s+a)$$

Ex.

$$\mathcal{L}\left\{te^{-2t}\right\} = \frac{1}{(s+2)^2}$$



# Example 3 (complex roots)

ODE with zero initial conditions (ICs)

$$\frac{d^2y(t)}{dt^2}+2\frac{dy(t)}{dt}+5y(t)=3u(t),\ \ y(0)=0,\ \ y'(0)=0$$

1. Laplace transform

$$s^2Y(s)+2sY(s)+5Y(s)=\frac{3}{s}$$

$$\Rightarrow Y(s)=\frac{3}{s(s^2+2s+5)}$$

*Complex roots*

# Example 3 (cont'd)

2. Partial fraction expansion

unknowns

$$Y(s) = \frac{3}{s(s^2+2s+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+5}$$

To obtain A, B & C:

$$3 = A(s^2+2s+5) + s(Bs+C)$$

$$\Rightarrow (A+B)s^2 + (2A+C)s + 5A - 3 = 0 \Rightarrow$$

$$A = \frac{3}{5}$$

$$B = -\frac{3}{5}$$

$$C = -\frac{6}{5}$$

# Example 3 (cont'd)

3. Inverse Laplace transform

$$Y(s) = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 5}$$

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{Bs+C}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{B(s+1)+C-B}{(s+1)^2+4}\right\} \\
&= B\mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+4}\right\} + \frac{C-B}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2+4}\right\} \\
&= Be^{-t}\cos 2t + \frac{C-B}{2}e^{-t}\sin 2t \\
\mathcal{L}^{-1}\{Y(s)\} &= \frac{3}{5} - \frac{3}{5}e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)
\end{aligned}$$

# Course roadmap

***Modeling***

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***Analysis***

- Time response
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  - Nyquist

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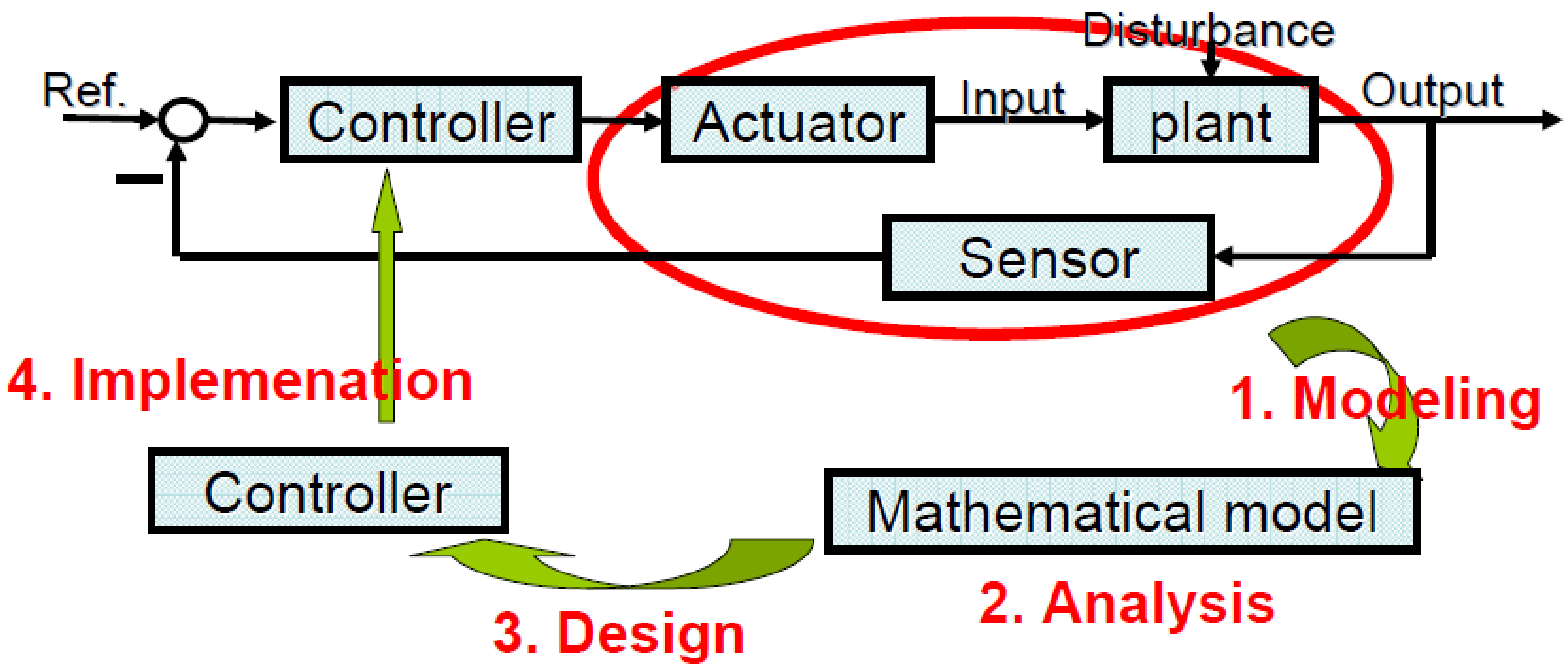
***Matlab simulations & laboratories***

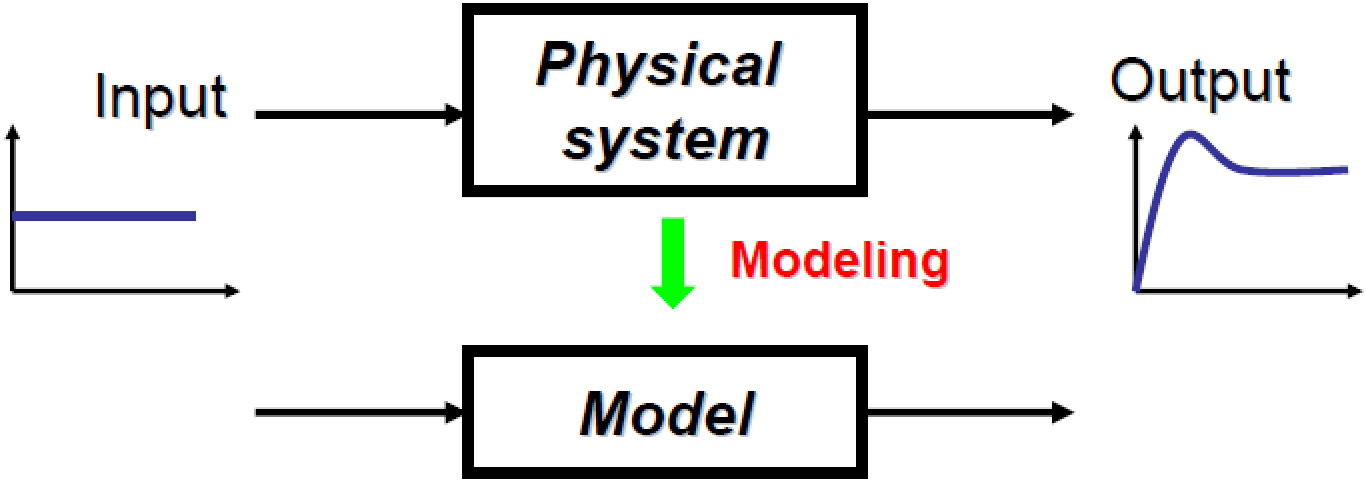
# Controller design procedure (review)




- What is the "mathematical model"?
- Transfer function
- Modeling of electrical & mechanical systems

# Mathematical model

- Representation of the input-output (signal) relation of a physical system




- A model is used for the **analysis** and **design** of control systems.

# Important remarks on models

- Modeling is the most important and difficult task in control system design.
- No mathematical model exactly represents a physical system.

$$\text{Math model} \neq \text{Physical system}$$
$$\text{Math model} \approx \text{Physical system}$$

- Do not confuse models with physical systems!
- In this course, we may use the term "system" to mean a mathematical model.

# Transfer function

- A transfer function is defined by

$$G(s) := \frac{Y(s)}{R(s)}$$

*Laplace transform of system output* (numerator $Y(s)$)

*Laplace transform of system input* (denominator $R(s)$)

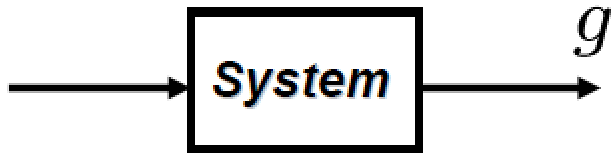
$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow \quad Y(s) = G(s)R(s)$$

- A system is assumed to be at rest. (Zero initial condition)

# Impulse response

- Suppose that u(t) is the unit impulse function and system is at rest.

$$r(t) = \delta(t)$$
$$R(s) = 1$$

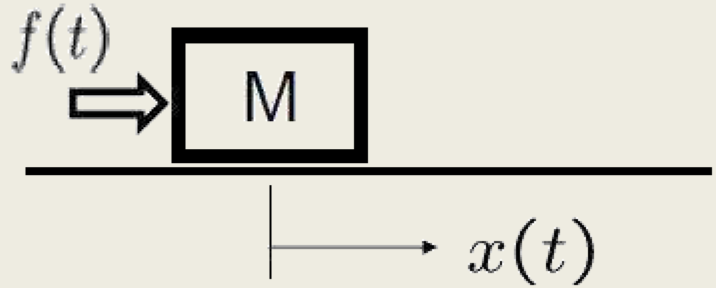



$$g(t)$$

- The output g(t) for the unit impulse input is called ***impulse response***.
- Since R(s)=1, the transfer function can also be defined as the Laplace transform of impulse response: $$G(s) := \mathcal{L}\{g(t)\}$$

# Example: Newton's law

$$M\frac{d^2x(t)}{dt^2} = f(t)$$




We want to know the trajectory of x(t). By Laplace transform,

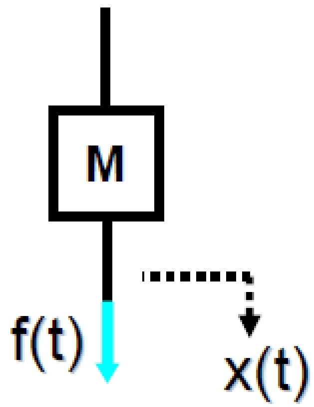
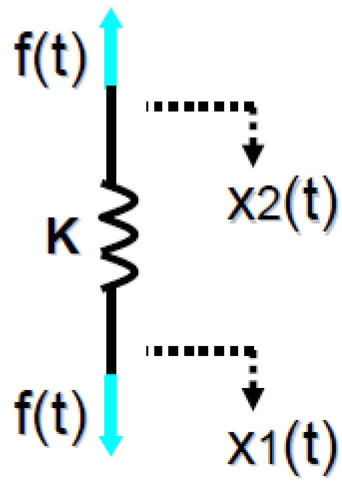
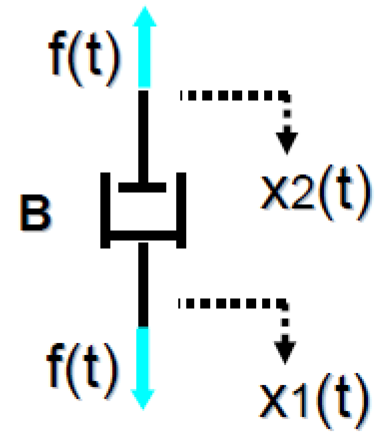
$$M\left(s^2X(s) - sx(0) - x'(0)\right) = F(s)$$

$$\Rightarrow\quad X(s) = \underbrace{\frac{1}{Ms^2}F(s)}_{} + \underbrace{\frac{x(0)}{s} + \frac{x'(0)}{s^2}}_{}$$

**(Total response) = (Forced response) + (Initial condition response)**

$$\Rightarrow\quad x(t) = \overbrace{\mathcal{L}^{-1}\left[\frac{1}{Ms^2}F(s)\right]}^{} + \overbrace{x(0)u(t) + x'(0)tu(t)}^{}$$

# Translational mechanical elements

| Mass | Spring | Damper |
|---|---|---|
|  |  |  |







**Mass**

$$f(t) = Mx''(t)$$

$$\Downarrow \quad \begin{pmatrix} x(0) = 0 \\ \dot{x}(0) = 0 \end{pmatrix}$$

$$F(s) = Ms^2X(s)$$

**Spring**

$$f(t) = K(x_1(t) - x_2(t))$$

$$\Downarrow$$

$$F(s) = K(X_1(s) - X_2(s))$$

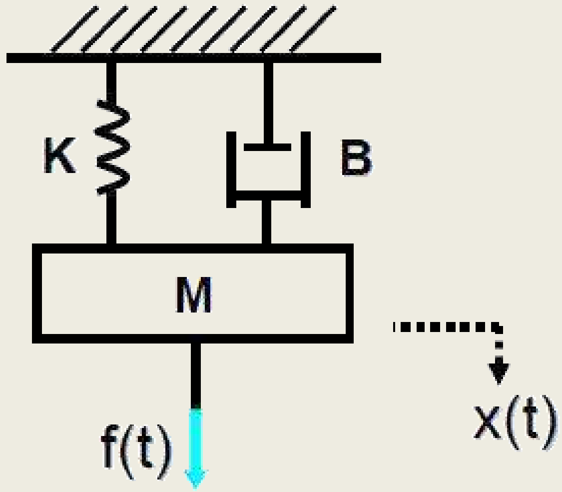
**Damper**

$$f(t) = B(x_1'(t) - x_2'(t))$$

$$\Downarrow \quad \begin{pmatrix} x_1(0) = 0 \\ x_2(0) = 0 \end{pmatrix}$$

$$F(s) = Bs(X_1(s) - X_2(s))$$

# Mass-spring-damper system




- Newton's law

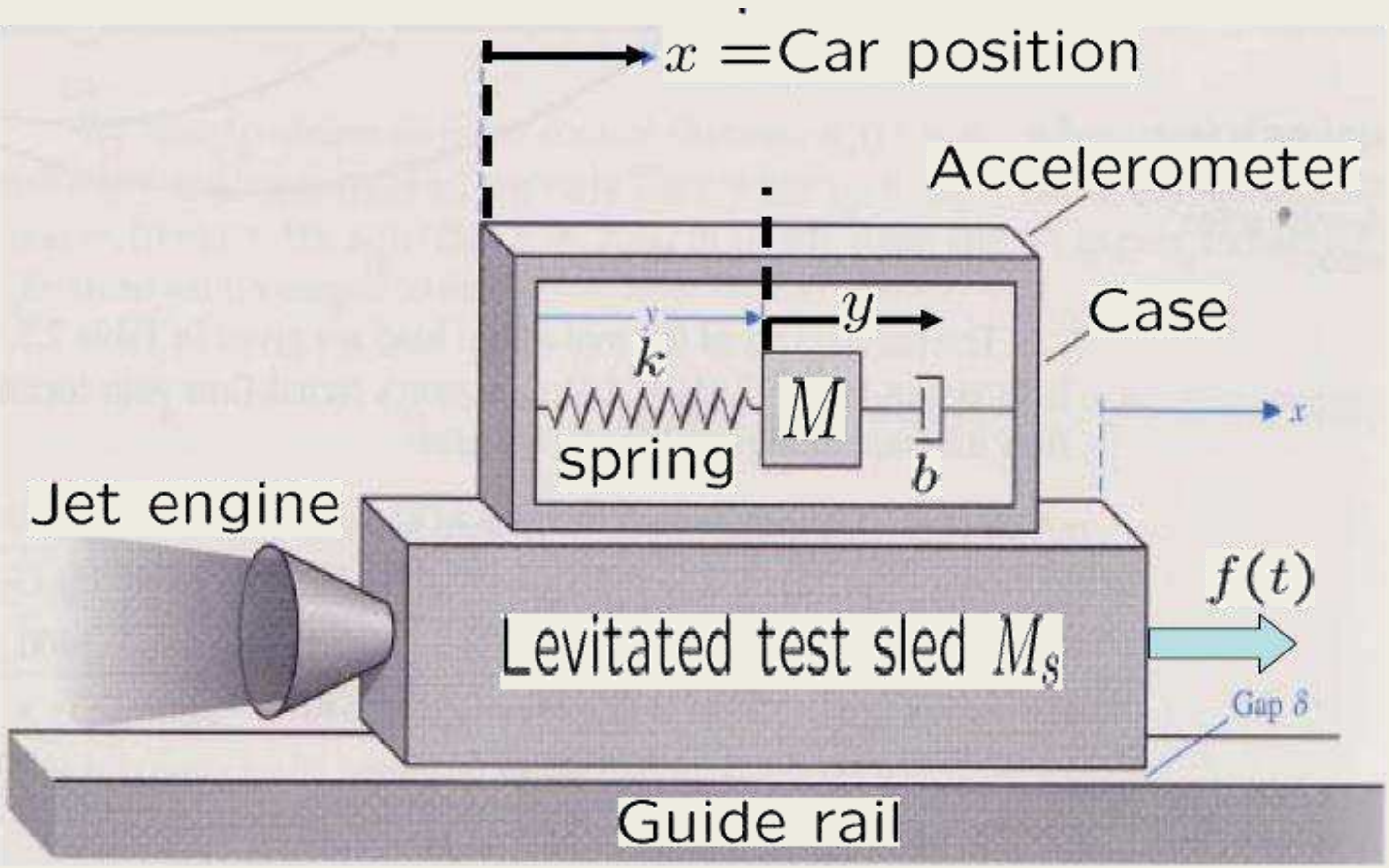
$$Mx''(t) = f(t) - Bx'(t) - Kx(t)$$

- By Laplace transform (with zero initial conditions),

$$X(s) = \frac{1}{Ms^2 + Bs + K} F(s)$$

*(2nd order system)*

# Ex: Mechanical accelerometer



# Ex: Accelerometer (cont'd)

- We would like to know how y(t) moves when unit step f(t) is applied with zero ICs.
- By Newton's law

$$\begin{cases} M\dfrac{d^2}{dt^2}(x(t)+y(t)) = -b\dfrac{dy(t)}{dt} - ky(t) \\ M_s\dfrac{d^2x(t)}{dt^2} = f(t) \end{cases}$$

$$\Rightarrow \quad My''(t) + by'(t) + ky(t) = -\frac{M}{M_s}f(t)$$

$$\overset{\mathcal{L}}{\Rightarrow} \quad Y(s) = -\frac{M}{M_s}\cdot\frac{1}{Ms^2+bs+k}\cdot\frac{1}{s} = -\frac{1}{M_s}\cdot\frac{1}{s^2+(b/M)s+(k/M)}\cdot\frac{1}{s}$$

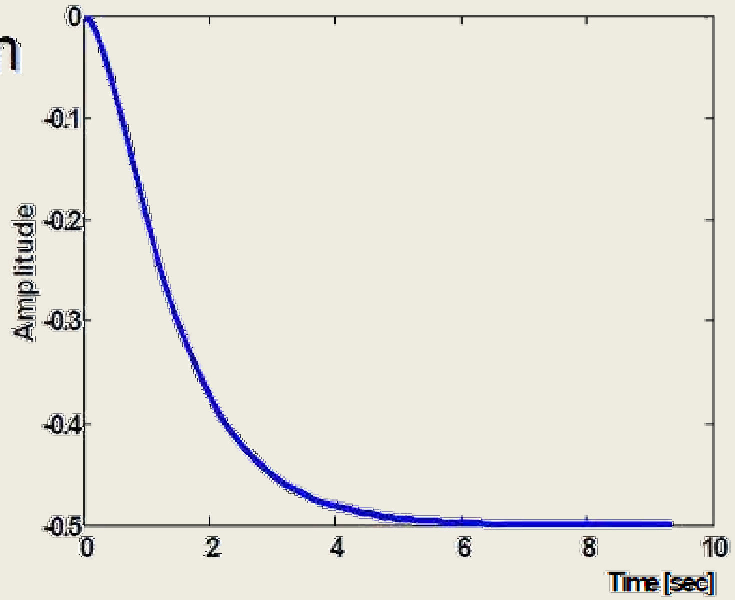
# Ex: Mechanical accelerometer (cont'd)

- Suppose that b/M=3, k/M=2 and Ms=1.
- Partial fraction expansion

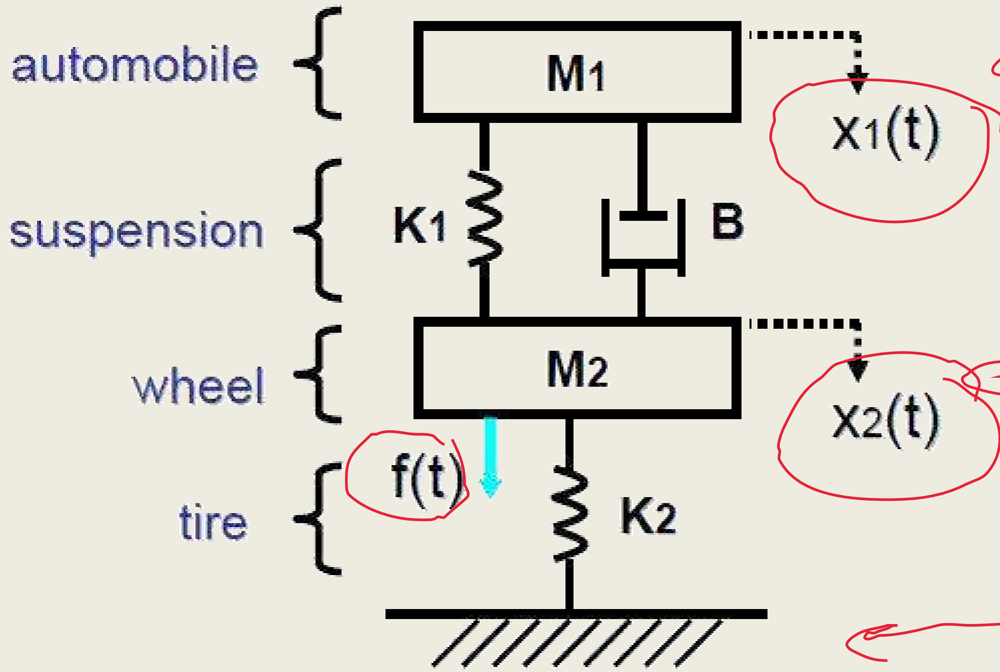
$$Y(s) = -\frac{1}{s^2+3s+2}\cdot\frac{1}{s} = -\frac{1}{2s}+\frac{1}{s+1}-\frac{1}{2(s+2)}$$

- Inverse Laplace transform

$$y(t) = \left(-\frac{1}{2}+e^{-t}-\frac{1}{2}e^{-2t}\right)u(t)$$



# Automobile suspension system




$$
\begin{cases}
M_1 x_1''(t) = -B(x_1'(t) - x_2'(t)) - K_1(x_1(t) - x_2(t)) \\
M_2 x_2''(t) = f(t) - B(x_2'(t) - x_1'(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t)
\end{cases}
$$

# Automobile suspension system

$$\begin{cases} M_1 x_1''(t) &= -B(x_1'(t) - x_2'(t)) - K_1(x_1(t) - x_2(t)) \\ M_2 x_2''(t) &= f(t) - B(x_2'(t) - x_1'(t)) - K_1(x_2(t) - x_1(t)) - K_2 x_2(t) \end{cases}$$

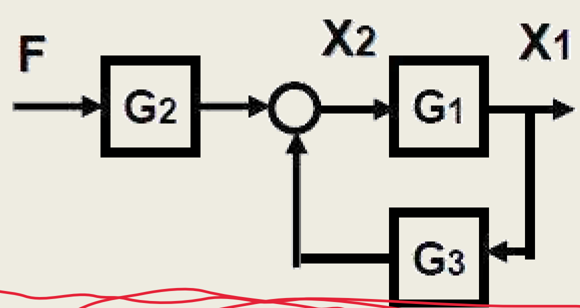
Laplace transform with zero ICs

$$\begin{cases} M_1 s^2 X_1(s) &= -B(sX_1(s) - sX_2(s)) - K_1(X_1(s) - X_2(s)) \\ M_2 s^2 X_2(s) &= F(s) - B(sX_2(s) - sX_1(s)) - K_1(X_2(s) - X_1(s)) - K_2 X_2(s) \end{cases}$$

$$\begin{cases} X_1(s) &= \underbrace{\dfrac{Bs + K_1}{M_1 s^2 + Bs + K_1}}_{G_1(s)} X_2(s) \\ X_2(s) &= \underbrace{\dfrac{1}{M_2 s^2 + Bs + K_1 + K_2}}_{G_2(s)} F(s) + \underbrace{\dfrac{Bs + K_1}{M_2 s^2 + Bs + K_1 + K_2}}_{G_3(s)} X_1(s) \end{cases}$$

# Automobile suspension (cont'd)

$$\begin{cases} X_1(s) = \underbrace{\dfrac{Bs+K_1}{M_1s^2+Bs+K_1}}_{G_1(s)} X_2(s) \\ X_2(s) = \underbrace{\dfrac{1}{M_2s^2+Bs+K_1+K_2}}_{G_2(s)} F(s) + \underbrace{\dfrac{Bs+K_1}{M_2s^2+Bs+K_1+K_2}}_{G_3(s)} X_1(s) \end{cases}$$




$$\frac{X_1(s)}{F(s)} = \frac{G_1(s)G_2(s)}{1 - G_1(s)G_3(s)}$$

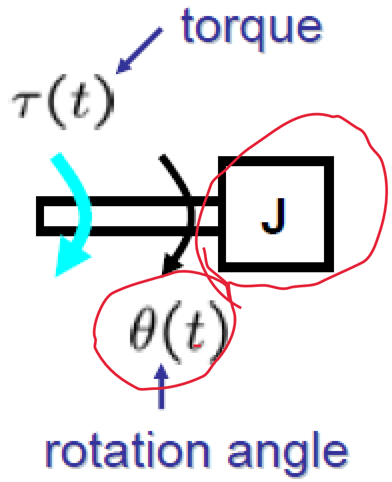
We will study how to derive this transfer function in the next lecture.

Block diagram
reduction rules

Umm Al-Qura University
College of Engineering & Islamic Architecture
Mechanical Engineering Department

# Rotational mechanical elements

## Moment of inertia

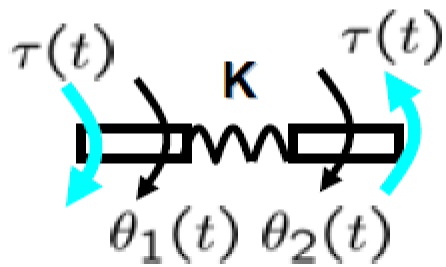



$$\tau(t) = J\theta''(t)$$

$$\begin{pmatrix} \theta(0) = 0 \\ \dot{\theta}(0) = 0 \end{pmatrix}$$

$$T(s) = Js^2\Theta(s)$$

## Rotational spring

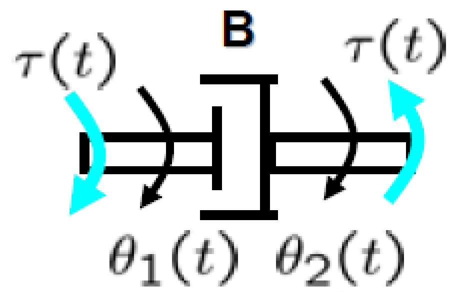



$$\boxed{\tau = K\theta}$$

$$\tau(t) = K(\theta_1(t) - \theta_2(t))$$

$$T(s) = K(\Theta_1(s) - \Theta_2(s))$$

## Friction



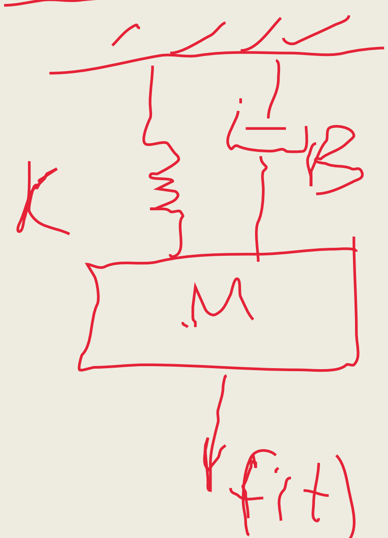

$$\boxed{\tau = B\dot{\theta}}$$

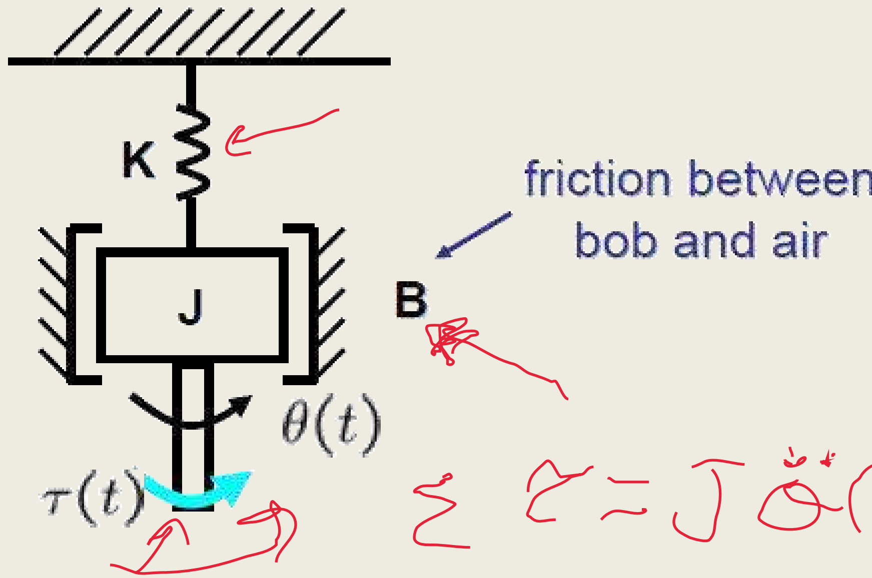
$$\tau(t) = B(\theta_1'(t) - \theta_2'(t))$$

$$\begin{pmatrix} \theta_1(0) = 0 \\ \theta_2(0) = 0 \end{pmatrix}$$

$$T(s) = Bs(\Theta_1(s) - \Theta_2(s))$$

# Torsional pendulum system




$\sum \tau = J\ddot{\theta}(t)$

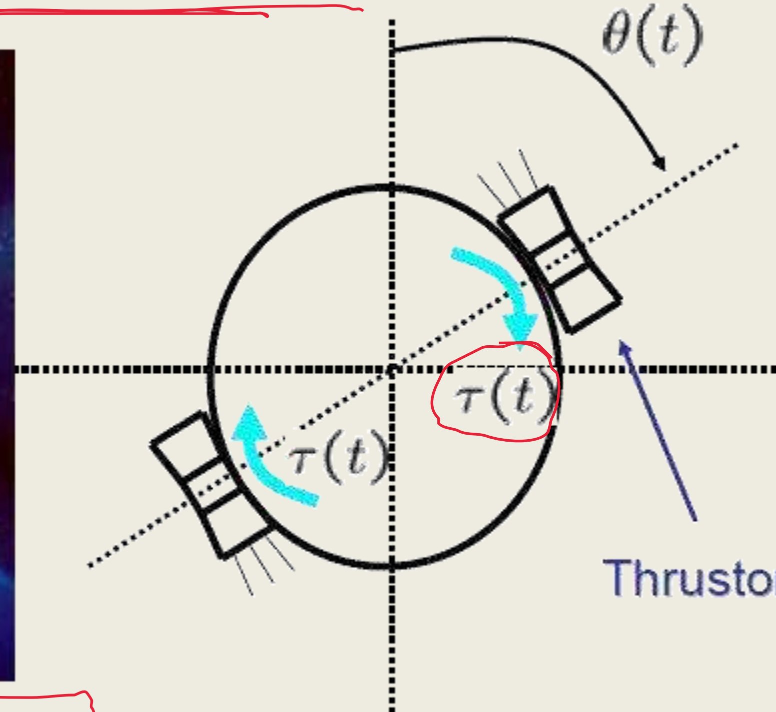
- Newton's law

$$J\theta''(t) = \tau(t) - B\theta'(t) - K\theta(t)$$

- By Laplace transform (with zero ICs),

$$G(s) = \boxed{\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + Bs + K}}$$ (2nd order system)

# Rigid satellite




- Broadcasting
- Weather forecast
- Communication
- GPS, etc.

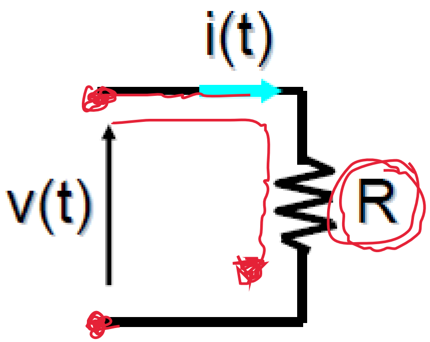
$$\boxed{\tau(t) = J\ddot{\theta}(t)}$$

$$\Rightarrow \quad G(s) = \boxed{\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2}}$$

*Double integrator*

# Models of electrical elements

## Resistance

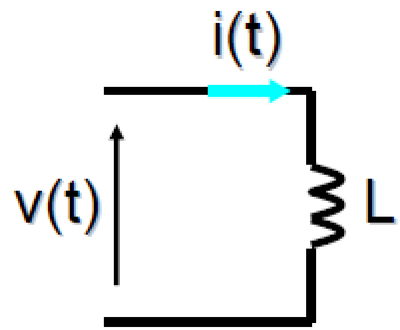



$$\boxed{v(t) = Ri(t)}$$

Laplace transform

$$\frac{V(s)}{I(s)} = R$$

## Inductance

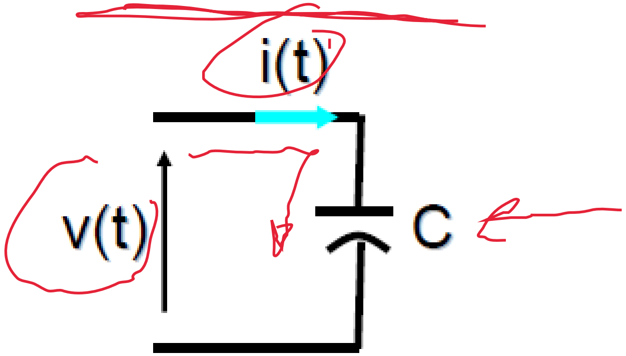



$$\boxed{v(t) = L\frac{di(t)}{dt}}$$

$(i(0) = 0)$

$$\frac{V(s)}{I(s)} = sL$$

## Capacitance

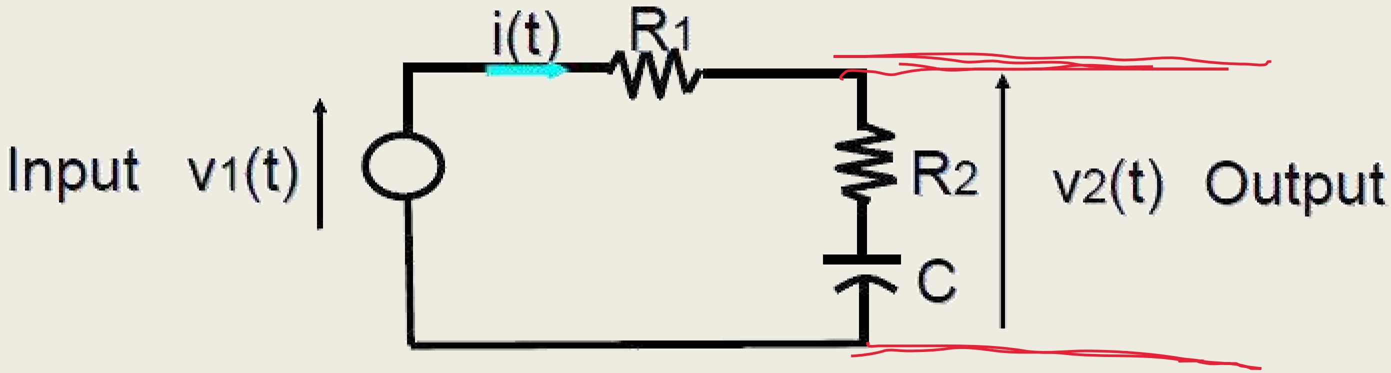



$$\boxed{i(t) = C\frac{dv(t)}{dt}}$$

$(v(0) = 0)$

$$\frac{V(s)}{I(s)} = \frac{1}{sC}$$

# Modeling example



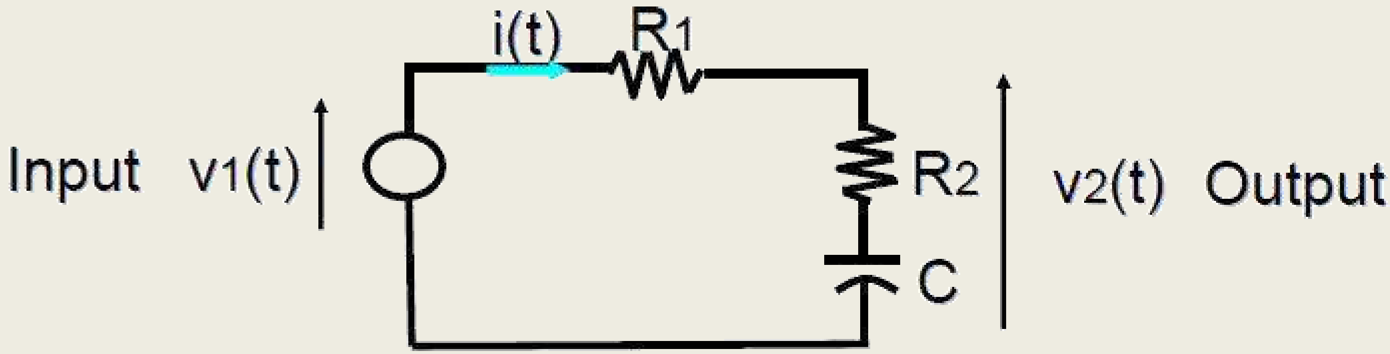

- Kirchhoff voltage law (with zero initial conditions)

$$\begin{aligned} v_1(t) &= (R_1 + R_2)i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \\ v_2(t) &= R_2 i(t) + \frac{1}{C}\int_0^t i(\tau)d\tau \end{aligned}$$

- By Laplace transform,

$$\begin{aligned} V_1(s) &= (R_1 + R_2)I(s) + \frac{1}{sC}I(s) \\ V_2(s) &= R_2 I(s) + \frac{1}{sC}I(s) \end{aligned}$$

# Modeling example (cont'd)




- Transfer function

$$G(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_2 + \frac{1}{sC}}{(R_1 + R_2) + \frac{1}{sC}}$$

$$= \frac{R_2 C s + 1}{(R_1 + R_2) C s + 1}$$

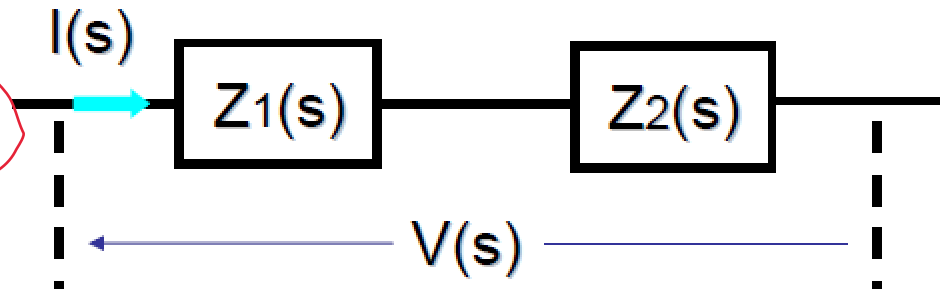
TF

(first-order system)

# Impedance computation

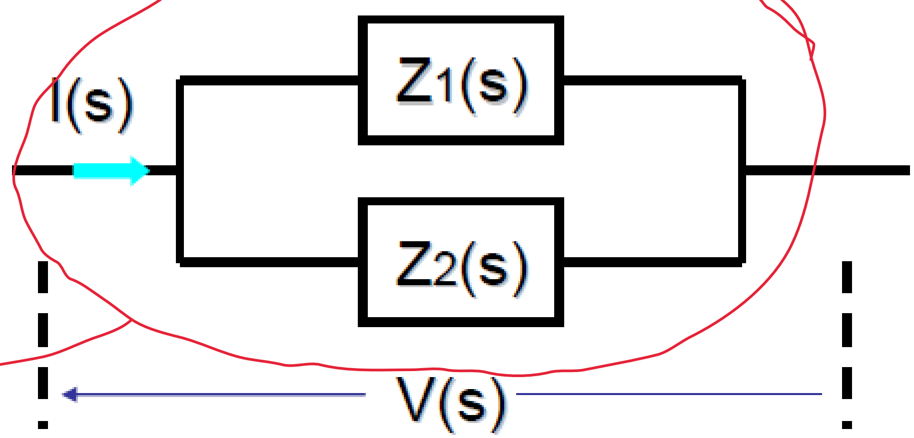
- Series connection

$$Z(s) = Z_1(s) + Z_2(s)$$




- Parallel connection

$$Z(s) = \frac{Z_1(s)Z_2(s)}{Z_1(s) + Z_2(s)}$$



# Summary

- Solution procedure to ODE's
  - Laplace transform
  - Partial fraction expansion
  - Inverse Laplace transform
- Modeling
  - Modeling is an important task
  - Mathematical model
  - Transfer function
  - Modeling of electrical and mechanical systems

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- Next
  - Modeling of electromechanical systems
  - Stability of linear control systems

# Assignment

ODE → Sol

**1)** Solve the differential equation

$$\frac{d^2x(t)}{dt^2} + 5\frac{dx(t)}{dt} + 4x(t) = 10u(t)$$

with the following initial conditions:

$$x(0) = \acute{x}(0) = 0$$

**2)** For the following $G(s)$, transform the relation $C(s) = G(s)R(s)$ into the differential equation.

AE

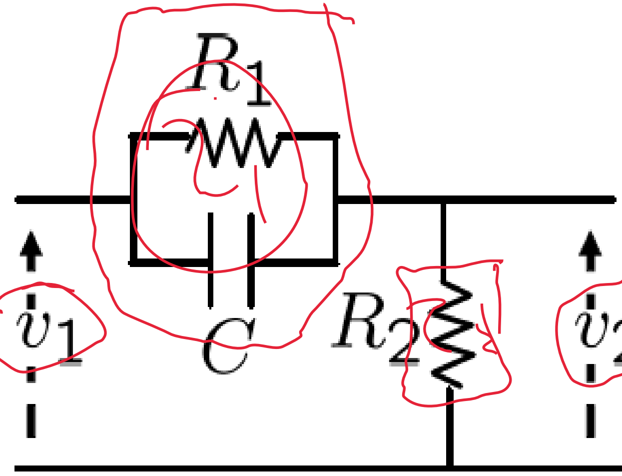
$$G(s) = \frac{60}{s^2 + 10s + 60}$$

$C(s) = \text{———} R(s)$

# Assignment

3) For the following circuit, find the transfer function from $V_1(s)$ to $V_2(s)$. If we input a constant voltage 10V, what is the steady state value of the output voltage?
(Hint: Use $v_1(t) = 10 \rightarrow V_1(s) = 10/s$ then find the final value theorem for $v_2(t)$)

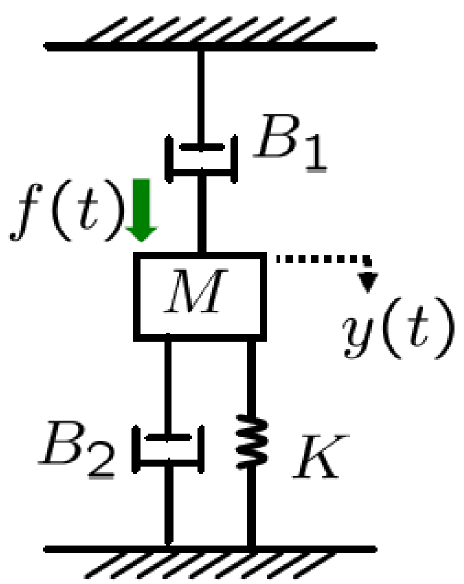
10V




? ?

# Assignment

**4)** For the following mechanical systems, obtain the transfer function from the input force $f$ to the mass displacement $y$



# End of Lecture 2